

Phenomenology of loop quantum cosmology

Mairi Sakellariadou

Department of Physics, King's College London, University of London, Strand WC2R 2LS, UK

E-mail: Mairi.Sakellariadou@kcl.ac.uk

Abstract. After introducing the basic ingredients of Loop Quantum Cosmology, I will briefly discuss some of its phenomenological aspects. Those can give some useful insight about the full Loop Quantum Gravity theory and provide an answer to some long-standing questions in early universe cosmology.

1. Introduction

The variety of precise astrophysical and cosmological data, available at present, combined with the large number of high energy physics experiments, are expected to give us the necessary ingredients to test fundamental theories and understand the very early phases of the evolution of our universe.

Cosmological inflation [1] remains the most promising candidate to solve the shortcomings of the standard hot big bang model, while it offers a causal explanation for the primordial fluctuations with the correct Cosmic Microwave Background (CMB) features. Despite this success, inflation suffers from a number of drawbacks [2]. In particular, compatibility between theory and measurements often necessitates fine-tuning of the inflationary parameters, inflation remains still a paradigm in search of a model, while it must prove itself generic [3, 4, 5, 6, 7, 8, 9].

There is undoubtedly an additional list of fundamental cosmological questions, still lacking a satisfying answer. One does not know, for instance, how close to the big bang a smooth space-time can be considered as the correct framework. Quantum gravity, a full theory which is supposed to resolve the big bang singularity, is still missing, while it is not known whether a new boundary condition is needed at the big bang, or whether quantum dynamical equations remain well-behaved at singularities. It is nevertheless clear that a smooth space-time background cannot be assumed as the correct description close to the big bang.

In a Hamiltonian formulation to a quantum theory, the absence of background metric indicates that Hamiltonian dynamics is generated by constraints. Physical states are solutions to quantum constraints implying that all physical laws are obtained from these solutions, while there is no external time according which evolution can be studied. One has to define a monotonic variable to play the rôle of emergent, or internal, time, and then build a framework within which the short-distance drawbacks of General Relativity near the big bang can be cured, maintaining however the agreement with General Relativity at large scales.

Loop Quantum Gravity (LQG) [10, 11] is a non-perturbative and background independent canonical quantisation of General Relativity in four space-time dimensions. Loop Quantum Cosmology (LQC) [12] is a cosmological mini-superspace model quantised with methods of LQG. The discreteness of spatial geometry and the simplicity of the setting allow for a complete

study of dynamics. The difference between LQC and other approaches of quantum cosmology, is that the input is motivated by a full quantum gravity theory. The simplicity of the setting, combined with the discreteness of spatial geometry provided by LQG, render feasible the overall study of LQC dynamics.

In what follows, I will briefly discuss some of the phenomenological consequences of LQC, a subject which gains constantly an increasing interest from the scientific community, due in particular to its successes.

2. Elements of LQG/LQC

Loop quantum cosmology is formulated in terms of $SU(2)$ holonomies of the connection and triads. The canonically conjugated variables are defined by the densitised triad E_i^a , and an $SU(2)$ valued connection A_a^i , where i refers to the Lie algebra index, and a is a spatial index with $a, i = 1, 2, 3$. The densitised triad gives information about spatial geometry (*i.e.*, the three metric), while the connection gives information about curvature (spatial and extrinsic one). Let me explain this: Quantum gravity introduces a discreteness to space-time. To quantise quantum gravity, this discreteness manifests itself as quanta of space. Since the quanta are three-dimensional, they can be characterised by a triad of numbers. The connection between adjacent pairs of these *quanta* form a two-dimensional surface, through which the geometric connection is defined. To ensure that local rotations do not induce different geometries, this connection must be $SO(3)$ invariant. By using connection-triad variables, arising from a canonical transformation of Arnowitt-Desner-Misner (variables, one can make an analogy with gauge theories, which is particularly useful when dealing with quantisation issues.

For any quantisation scheme based on a Hamiltonian framework (*e.g.*, LQC) or an action principle (*e.g.*, path integral) for a homogeneous and flat model, divergences appear which need to be regularised. To remove the divergences that occur in non-compact topologies, the spatial homogeneity and Hamiltonian are restricted to a finite fiducial cell, of scale μ_0 , with finite volume $V_0 = \int d^3x \sqrt{|{}^0q|}$, where 0q is the determinant of the fiducial background metric.

In the case of a spatially flat background, derived from the Bianchi I model, the isotropic connection can be expressed in terms of the dynamical component of the connection $\tilde{c}(t)$ as

$$A_a^i = \tilde{c}(t)\omega_a^i, \quad (1)$$

with ω_a^i a basis of left-invariant one-forms $\omega_a^i = dx^i$. The densitised triad can be decomposed using the Bianchi I basis vector fields $X_i^a = \delta_i^a$ as

$$E_i^a = \sqrt{{}^0q}\tilde{p}(t)X_i^a, \quad (2)$$

where 0q stands for the determinant of the fiducial background metric ${}^0q_{ab} = \omega_a^i\omega_{bi}$, and $\tilde{p}(t)$ denotes the remaining dynamical quantity after symmetry reduction.

In terms of the metric variables with three-metric $q_{ab} = a^2\omega_a^i\omega_{bi}$, the dynamical quantity is just the scale factor $a(t)$. Given that the Bianchi I basis vectors are $X_i^a = \delta_i^a$,

$$|\tilde{p}| = a^2, \quad (3)$$

where the absolute value is taken because the triad has an orientation. Since the basis vector fields are spatially constant in the spatially flat model, the connection component is

$$\tilde{c} = \text{sgn}(\tilde{p})\gamma\frac{\dot{a}}{N}; \quad (4)$$

N is the lapse function and γ the Barbero-Immirzi parameter, a quantum ambiguity parameter approximately equal to 0.2375.

The canonical variables \tilde{c}, \tilde{p} are related through the Poisson bracket

$$\{\tilde{c}, \tilde{p}\} = \frac{\kappa\gamma}{3} V_0 , \quad (5)$$

where V_0 the volume of the elementary cell adapted to the fiducial triad and $\kappa \equiv 8\pi G$.

Defining the triad component p , determining the physical volume of the fiducial cell, and the connection component c , determining the rate of change of the physical edge length of the fiducial cell, as

$$p = V_0^{2/3} \tilde{p} , \quad c = V_0^{1/3} \tilde{c} , \quad (6)$$

respectively, we obtain

$$\{c, p\} = \frac{\kappa\gamma}{3} , \quad (7)$$

independent of the volume V_0 of the fiducial cell.

To quantise the theory one faces the usual difficulty of quantum cosmology. Namely, the metric itself has to be considered as a physical field, thus it must be quantised; it is not a fixed background. Any *standard* quantisation scheme fails: Fock quantisation fails, while even free Hamiltonians are non-quadratic in metric dependence, so there is no simple perturbation analysis. The way out is to use gauge theory variables to define holonomies of the connection along a given edge

$$h_e(A) = \mathcal{P} \exp \int ds \dot{\gamma}^\mu(s) A_\mu^i(\gamma(s)) \tau_i , \quad (8)$$

where \mathcal{P} indicates a path ordering of the exponential, γ^μ is a vector tangent to the edge and $\tau_i = -i\sigma_i/2$, with σ_i the Pauli spin matrices, and fluxes of a triad along an S surface

$$E(S, f) = \int_S \epsilon_{abc} E^{ci} f_i dx^a dx^b , \quad (9)$$

with f_i an $SU(2)$ valued test function. Certainly, at the level of LQC these variables may seem ad-hoc and unnatural, however their motivation follows naturally from the full LQG theory.

Thus, the basic configuration variables in LQC are holonomies of the connection

$$h_i^{(\mu_0)}(A) = \cos\left(\frac{\mu_0 c}{2}\right) \mathbf{1} + 2 \sin\left(\frac{\mu_0 c}{2}\right) \tau_i , \quad (10)$$

along a line segment $\mu_0 {}^0e_i^a$ and the flux of the triad

$$F_S(E, f) \propto p ,$$

where the basic momentum variable is the triad component p , $\mathbf{1}$ is the identity 2×2 matrix and $\tau_i = -i\sigma_i/2$ is a basis in the Lie algebra $SU(2)$ satisfying the relation

$$\tau_i \tau_j = (1/2) \epsilon_{ijk} \tau^k - (1/4) \delta_{ij} .$$

In the classical theory, curvature can be expressed as a limit of the holonomies around a loop as the area enclosed by the loop shrinks to zero. In quantum geometry however, the loop cannot be continuously shrunk to zero area and the eigenvalues of the area operator are discrete. Thus, there is a smallest non-zero eigenvalue, the area gap Δ [13]. As a result, the Wheeler-de Witt differential equation gets replaced by a difference equation whose step size is controlled by Δ .

Let me start with the *old* quantisation procedure. Along the lines of LQG, one considers $e^{i\mu_0 c/2}$ (with μ_0 an arbitrary real number) and p , as the elementary classical variables, which

have well-defined analogues. Using the Dirac bra-ket notation and setting $e^{i\mu_0 c/2} = \langle c|\mu\rangle$, the action of the operator \hat{p} acting on the basis states $|\mu\rangle$ reads

$$\hat{p}|\mu\rangle = \frac{\kappa\gamma\hbar|\mu|}{6}|\mu\rangle, \quad (11)$$

where μ (a real number) stands for the eigenstates of \hat{p} , satisfying the orthonormality relation $\langle\mu_1|\mu_2\rangle = \delta_{\mu_1,\mu_2}$. The action of the $\widehat{\exp\left[\frac{i\mu_0}{2}c\right]}$ operator acting on basis states $|\mu\rangle$ is

$$\widehat{\exp\left[\frac{i\mu_0}{2}c\right]}|\mu\rangle = \exp\left[\mu_0\frac{d}{d\mu}\right]|\mu\rangle = |\mu + \mu_0\rangle, \quad (12)$$

where μ_0 is any real number. Thus, in the *old* quantisation, the operator $e^{i\mu_0 c/2}$ acts as a simple shift operator. As in the full LQG theory, there is no operator corresponding to the connection, however the action of its holonomy is well defined. The action of the holonomies, $\hat{h}_i^{(\mu_0)}$, of the gravitational connection on the basis states is given by [14]

$$\hat{h}_i^{(\mu_0)}|\mu\rangle = (\widehat{\text{cs}}\mathbf{1} + 2\widehat{\text{sn}}\tau_i)|\mu\rangle, \quad (13)$$

where,

$$\begin{aligned} \widehat{\text{cs}}|\mu\rangle &\equiv \widehat{\cos(\mu_0 c/2)}|\mu\rangle = [|\mu + \mu_0\rangle + |\mu - \mu_0\rangle]/2, \\ \widehat{\text{sn}}|\mu\rangle &\equiv \widehat{\sin(\mu_0 c/2)}|\mu\rangle = [|\mu + \mu_0\rangle - |\mu - \mu_0\rangle]/(2i). \end{aligned} \quad (14)$$

The gravitational part of the Hamiltonian operator in terms of SU(2) holonomies and the triad component, in the irreducible $J = 1/2$ representation, reads [14]

$$\hat{\mathcal{C}}_g = \frac{2i}{\kappa^2\hbar\gamma^3\mu_0^3} \text{tr} \sum_{ijk} \epsilon^{ijk} \left(\hat{h}_i^{(\mu_0)} \hat{h}_j^{(\mu_0)} \hat{h}_i^{(\mu_0)-1} \hat{h}_j^{(\mu_0)-1} \hat{h}_k^{(\mu_0)} \left[\hat{h}_k^{(\mu_0)-1}, \hat{V} \right] \right) \text{sgn}(\hat{p}). \quad (15)$$

The action of the self-adjoint Hamiltonian constraint operator, $\hat{\mathcal{H}}_g = (\hat{\mathcal{C}}_g + \hat{\mathcal{C}}_g^\dagger)/2$ on the basis states $|\mu\rangle$ is

$$\hat{\mathcal{H}}_g|\mu\rangle = \frac{3}{4\kappa^2\gamma^3\hbar\mu_0^3} \left\{ [R(\mu) + R(\mu + 4\mu_0)]|\mu + 4\mu_0\rangle - 4R(\mu)|\mu\rangle + [R(\mu) + R(\mu - 4\mu_0)]|\mu - 4\mu_0\rangle \right\}, \quad (16)$$

where

$$R(\mu) = (\kappa\gamma\hbar/6)^{3/2} \left| |\mu + \mu_0|^{3/2} - |\mu - \mu_0|^{3/2} \right|. \quad (17)$$

The dynamics are then determined by the Hamiltonian constraint

$$(\hat{\mathcal{H}}_g + \hat{\mathcal{H}}_\phi)|\Psi\rangle = 0, \quad (18)$$

where $\hat{\mathcal{H}}_\phi$ stands for the matter Hamiltonian. In the full LQG theory there is an infinite number of constraints, while in LQC there is only one integrated Hamiltonian constraint. Matter is introduced by just adding the actions of matter components to the gravitational action. We can finally obtain difference equations analogous to the differential Wheeler-de Witt equations.

More precisely, the constraint equation on the physical wave-functions $|\Psi\rangle$, which can be expanded using the basis states as $|\Psi\rangle = \sum_{\mu} \Psi_{\mu}(\phi)|\mu\rangle$ with summation over values of μ and where the dependence of the coefficients on ϕ represents the matter degrees of freedom, reads [15]

$$\left[\left| V_{\mu+5\mu_0} - V_{\mu+3\mu_0} \right| + \left| V_{\mu+\mu_0} - V_{\mu-\mu_0} \right| \right] \Psi_{\mu+4\mu_0}(\phi) - 4 \left| V_{\mu+\mu_0} V_{\mu-\mu_0} \right| \Psi_{\mu}(\phi) + \left[\left| V_{\mu-3\mu_0} - V_{\mu-5\mu_0} \right| + \left| V_{\mu+\mu_0} - V_{\mu-\mu_0} \right| \right] \Psi_{\mu-4\mu_0}(\phi) = -\frac{4\kappa^2\gamma^3\hbar\mu_0^3}{3} \mathcal{H}_{\phi}(\mu) \Psi_{\mu}(\phi) ; \quad (19)$$

the matter Hamiltonian $\hat{\mathcal{H}}_{\phi}$ is assumed to act diagonally on the basis states with eigenvalues $\mathcal{H}_{\phi}(\mu)$. Equation (19) is the quantum evolution (in internal time μ) equation; there is no continuous variable (the scale factor in classical cosmology), but a label μ with discrete steps. The wave-function $\Psi_{\mu}(\phi)$, depending on internal time μ and matter fields ϕ , determines the dependence of matter fields on the evolution of the universe, with a massless scalar field playing the rôle of emergent time. Thus, in LQC the quantum evolution is governed by a second order difference equation, rather than the second order differential equation of the Wheeler-de Witt quantum cosmology. As the universe becomes large and enters the semi-classical regime, its evolution can be approximated by the differential Wheeler-de Witt equation.

3. Lattice refinement

The quantised holonomies were at first assumed to be shift operators with a fixed magnitude, leading to a quantised Hamiltonian constraint being a difference equation with a constant interval between points on the lattice. While these models can be used to study certain aspects of the quantum regime, they were found to lead to serious instabilities in the continuum, semi-classical limit [16, 17]. Considering the continuum limit of the Hamiltonian constraint operator, we have so far assumed that Ψ does not vary much on scales of the order of $4\mu_0$ (known as *pre-classicality*), so that one can smoothly interpolate between the points on the discrete function $\Psi_{\mu}(\phi)$ and approximate them by the continuous function $\Psi(\mu, \phi)$. Under this assumption, the difference equation can be very well approximated by a differential equation for a continuous wave-function. However, the form of the wave-function indicates that the period of oscillations can decrease as the scale increases. Thus, the assumption of *pre-classicality* can break down at large scales, leading to deviations from the classical behaviour.

Let me explain this point a bit further: In the underlying full LQG theory, the contributions to the (discrete) Hamiltonian operator depend on the state which describes the universe. As the volume grows (i.e., the universe expands), the number of contributions increases. Thus, the Hamiltonian constraint operator is expected to create new vertices of a lattice state (in addition to changing their edge labels), which in LQC results in a refinement of the discrete lattice [18]. Lattice refinement is also required from phenomenological reasons [17, 19]; for instance, it renders a successful inflationary era more natural [19]. The effect of lattice refinement has been modelled and the elimination of the instabilities in the continuum era has been explicitly shown.

The appropriate lattice refinement model should be obtained from the full LQG theory. In principle, one should use the full Hamiltonian constraint and find the way that its action balances the creation of new vertices as the volume increases. Instead, phenomenological arguments have been used, where the choice of the lattice refinement model is constrained by the form of the matter Hamiltonian [20]. In particular, LQC can generically support inflation, and other matter fields, without the onset of large scale quantum gravity corrections, only for a particular model of lattice refinement [20]. This choice is the only one for which physical quantities are independent of the elementary cell adopted to regulate spatial integrations [21], and moreover, it is exactly the choice required for the uniqueness of the factor ordering of the Wheeler-de Witt equation [22].

Allowing the length scale of the holonomies to vary dynamically, the form of the difference equation, describing the evolution, changes. Since the parameter μ_0 determines the step-size of the difference equation, assuming the lattice size is growing, the step-size of the difference equation is not constant in the original triad variables. Let us consider the particular case of

$$\mu_0 \rightarrow \tilde{\mu}(\mu) = \mu_0 \mu^{-1/2} , \quad (20)$$

suggested by certain intuitive heuristic approaches, such as noting that the minimum area used to regulate the holonomies should be a physical area [23], or that the discrete step size of the difference equation should always be of the order of the Planck volume. Moreover, this choice also results in a significant simplification of the difference equation, compared to more general lattice refinement schemes. The basic operators are given by replacing μ_0 with $\tilde{\mu}$. Upon quantisation [23]

$$\widehat{e^{i\tilde{\mu}c/2}}|\mu\rangle = e^{-i\tilde{\mu}\frac{d}{d\mu}}|\mu\rangle , \quad (21)$$

which is no longer a simple shift operator since $\tilde{\mu}$ is a function of μ . Changing the basis to

$$\nu = \mu_0 \int \frac{d\mu}{\tilde{\mu}} = \frac{2}{3} \mu^{3/2} , \quad (22)$$

one gets

$$e^{-i\tilde{\mu}\frac{d}{d\mu}}|\nu\rangle = e^{-i\mu_0\frac{d}{d\nu}}|\nu\rangle = |\nu + \mu_0\rangle . \quad (23)$$

Thus, in the new variables the holonomies act as simple shift operators, with parameter length μ_0 . In this sense, the basis $|\nu\rangle$ is a much more natural choice than $|\mu\rangle$. One can then proceed as in the previous case of a fixed spatial lattice and write down the Hamiltonian constraint.

3.1. Constraints on inflation

Let me consider first a fixed and then a dynamically varying lattice and solve the second order difference equation in the continuum limit. Two constraints can be imposed on the inflaton potential: the first one so that the continuum approximation is valid, and the second one so that there is agreement with the CMB measurements on large angular scales. A combination of the two constraints in the context of a particular inflationary model, will give us [20] the conditions for natural and successful inflation within LQC.

More precisely, let us separate the wave-function $\Psi(p, \phi)$ into $\Psi(p, \phi) = \Upsilon(p)\Phi(\phi)$ and approximate the dynamics of the inflaton field, ϕ , by setting $V(\phi) = V_\phi p^{\delta-3/2}$, where V_ϕ is a constant and $\delta = 3/2$ in the case of slow-roll, to get [20]

$$p^{-1/2} \frac{d}{dp} \left[p^{-1/2} \frac{d}{dp} \left(p^{3/2} \Upsilon(p) \right) \right] + \beta V_\phi p^\delta \Upsilon(p) = 0 , \quad (24)$$

with solutions [20]

$$\Upsilon(p) \approx p^{-(9+2\delta)/8} \sqrt{\frac{2\delta+3}{2\sqrt{\beta V_\phi} \pi}} \left[C_1 \cos \left(x - \frac{3\pi}{2(2\delta+3)} - \frac{\pi}{4} \right) + C_2 \sin \left(x - \frac{3\pi}{2(2\delta+3)} - \frac{\pi}{4} \right) \right] , \quad (25)$$

where $x = 4\sqrt{\beta V_\phi} (2\delta+3)^{-1} p^{(2\delta+3)/4}$ and $\beta = 96/(\kappa \hbar^2)$.

Without lattice refinement, the discrete nature of the underlying lattice would eventually be unable to support the oscillations and the assumption of *pre-classicality* will break down, implying that the discrete nature of space-time becomes significant on large scales. For the end of inflation to be describable using classical General Relativity, it must end before a scale, at

which the assumption of *pre-classicality* breaks down and the semi-classical description is no longer valid, is reached. Let me quantify this constraint: The separation between two successive zeros of $\Upsilon(p)$ is

$$\Delta p = \frac{\pi}{\sqrt{\beta V_\phi}} p^{(1-2\delta)/4} . \quad (26)$$

For the continuum limit to be valid, the wave-function must vary slowly on scales of the order of $\mu_c = 4\tilde{\mu}$, leading to [20]

$$\Delta p > 4\mu_0 \left(\frac{\kappa\gamma\hbar}{6} \right)^{3/2} p^{-1/2} , \quad (27)$$

which implies [20]

$$V_\phi < \frac{27\pi^2}{192\mu_0^2\gamma^3\kappa^2\hbar} p^{(3-2\delta)/2} . \quad (28)$$

Assuming slow-roll inflation, we set $\delta \approx 3/2$. Setting $\mu_0 = 3\sqrt{3}/2$, the constraint on the inflationary potential, in units of $\hbar = 1$, reads

$$V(\phi) \lesssim 2.35 \times 10^{-2} l_{\text{Pl}}^{-4} , \quad (29)$$

which is a weaker constraint than the one imposed for fixed lattices, namely [20]

$$V_\phi \ll 10^{-28} l_{\text{Pl}}^{-4} , \quad (30)$$

assuming that half of the inflationary era takes place during the classical era.

One can further constrain the inflationary potential so that the fractional over-density in Fourier space and at horizon crossing is consistent with the COBE-DMR measurements, namely

$$\frac{[V(\phi)]^{3/2}}{V'(\phi)} \approx 5.2 \times 10^{-4} M_{\text{Pl}}^3 . \quad (31)$$

To do so, one must however adopt a particular inflationary model. As such, let us select $V(\phi) = m^2\phi^2/2$. Combining the two constraints we obtain [20]

$$m \lesssim 70(e^{-2N_{\text{cl}}}) M_{\text{Pl}} \quad \text{and} \quad m \lesssim 10 M_{\text{Pl}} , \quad (32)$$

for the fixed and varying lattices, respectively.

In conclusion, the requirement that a significant proportion of a successful (in particular with respect to the CMB measurements) inflationary regime takes place during the classical era, imposes a strong constraint on the inflaton mass. Such a constraint turns out however to be much softer, and therefore rather natural, once lattice refinement is considered. In this sense, we argue [20] that lattice refinement is essential to achieve a successful inflationary era, provided inflation can be generically (*i.e.*, with generic initial conditions) realised.

3.2. Constraints on the matter Hamiltonian

Given a lattice refinement model, we will show [19] that only certain types of matter can be allowed. To do so, we parametrise the lattice refinement by A and the matter Hamiltonian by δ and solve the Hamiltonian constraint. The restrictions on the two-dimensional parameter space will become apparent once physical restrictions to the solutions of the wave-functions are imposed [19].

To be more precise, consider

$$\tilde{\mu} = \mu_0 \mu^A , \quad (33)$$

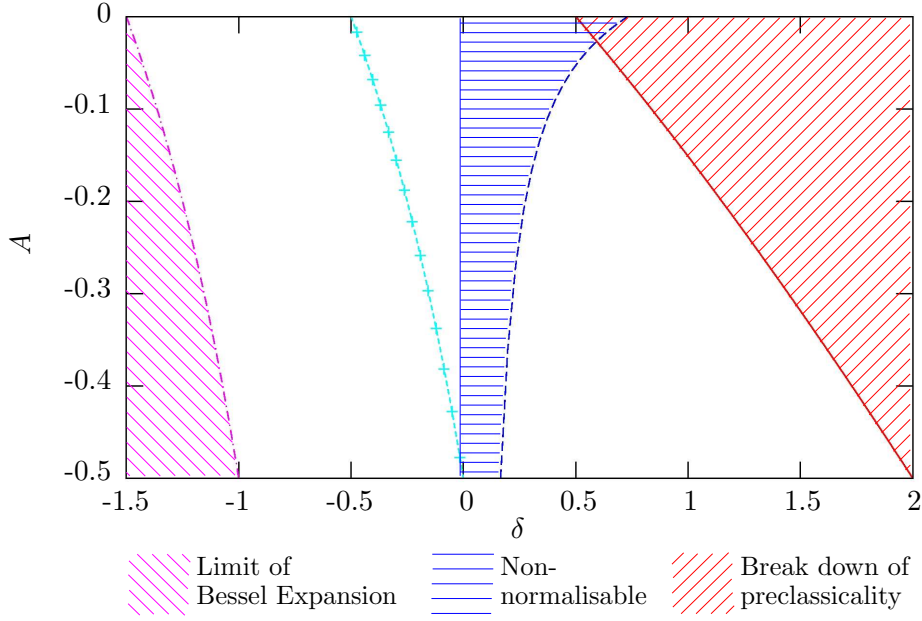


Figure 1. The allowed types of matter content are significantly restricted. For a varying lattice ($A \neq 0$) it is not always possible to treat the large scale behaviour of the wave-functions perturbatively (dashed line with crosses) [19].

leading to

$$\nu = \frac{\tilde{\mu}_0 \mu^{1-A}}{\mu_0(1-A)} . \quad (34)$$

Being only interested in the large scale limit, we approximate [19] the matter Hamiltonian by $\hat{\mathcal{H}}_\phi = \hat{\nu}^\delta \hat{\epsilon}(\phi)$, implying

$$\hat{\epsilon}(\phi) \Psi \equiv \epsilon(\phi) \Psi = -\nu^{-\delta} \hat{\mathcal{H}}_g \Psi . \quad (35)$$

A necessary condition so that the wave-functions are physical, is that the finite norm of the physical wave-functions, defined by $\int_{\phi=\phi_0} d\nu |\nu|^\delta \bar{\Psi}_1 \Psi_2$, is independent of the choice of $\phi = \phi_0$. The solutions of the constraint are renormalisable provided they decay, on large scales, faster than $\nu^{-1/(2\delta)}$.

To solve the constraint equation, we need to specify the form of \mathcal{H}_ϕ , which has in general two terms with different scale dependence. Being interested in the large scale limit, there is one dominant term, allowing to write [19]

$$\beta \mathcal{H}_\phi = \epsilon_\nu(\phi) \nu^{\delta_\nu} , \quad (36)$$

where the function ϵ_ν is constant with respect to ν .

Solving [19] the constraint equation, we only consider the physical solutions. The large scale behaviour of the wave-functions must be normalisable, which is a necessary condition for having physical wave-functions, while the wave-functions should preserve *pre-classicality* at large scales, which is a necessary condition for the validity of the continuum limit. We thus obtain [19] constraints to the two-dimensional parameter space (A, δ) , shown in Fig. 1 for A in the range $0 < A < -1/2$, imposed from full LQG theory considerations [17].

In conclusion, the continuum limit of the Hamiltonian constraint equation is sensitive to the choice of the lattice refinement model and only a limited range of matter components can be supported within a particular choice.

3.3. Uniqueness of the factor ordering in the Wheeler-de Witt equation

I will now show that the lattice refinement model $\tilde{\mu} = \mu_0 \mu^{-1/2}$, argued [22] to be the only one achieved by both physical considerations of large scale physics and consistency of the quantisation structure, is also the only model which makes the factor ordering ambiguities of LQC to disappear in the continuum limit [22].

Let me be more specific. Writing the gravitational part of the Hamiltonian constraint in terms of the triad and the holonomies of the connection, one realises that there are many ways of doing so. Considering for example,

$$\hat{\mathcal{C}}_g = \frac{2i}{\kappa^2 \hbar \gamma^3 k^3} \text{tr} \sum_{ijk} \epsilon^{ijk} \left(\hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k \left[\hat{h}_k^{-1}, \hat{V} \right] \right), \quad (37)$$

there are many possible choices of factor ordering that could have been made at this point; classically, the actions of the holonomies commute. However, each of these factor ordering choices leads to a different factor ordering of the Wheeler-de Witt equation in the continuum limit. The action of the factor ordering chosen in Eq. (37) leads [22] to

$$\epsilon_{ijk} \text{tr} \left(\hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k \left[\hat{h}_k^{-1}, \hat{V} \right] \right) = -24 \hat{s} \hat{n}^2 \hat{c} s^2 \left(\hat{c} \hat{s} \hat{V} \hat{s} \hat{n} - \hat{s} \hat{n} \hat{V} \hat{c} s \right), \quad (38)$$

while other choices have certainly a different action.

Defining $\hat{V}|\nu\rangle = V_\nu|\nu\rangle$, with \hat{V} the volume operator with eigenvalues V_ν , the action of the above factor ordering on a general state $|\Psi\rangle = \sum_\nu \psi_\nu |\nu\rangle$ in the Hilbert space, reads [22]

$$\begin{aligned} \epsilon_{ijk} \text{tr} \left(\hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k \left[\hat{h}_k^{-1}, \hat{V} \right] \right) |\Psi\rangle &= \frac{-3i}{4} \sum_\nu \left[\left(V_{\nu-3k} - V_{\nu-5k} \right) \psi_{\nu-4k} - 2 \left(V_{\nu+k} - V_{\nu-k} \right) \psi_\nu \right. \\ &\quad \left. + \left(V_{\nu+5k} - V_{\nu+3k} \right) \psi_{\nu+4k} \right] |\nu\rangle; \end{aligned} \quad (39)$$

the action of any other factor ordering choice can be obtained in a similar manner. By noting that the volume is given by

$$V_\nu |\nu\rangle \sim [\mu(\nu)]^{3/2} |\nu\rangle, \quad (40)$$

where $\mu(\nu)$ is obtained by

$$\nu = \frac{k \mu^{1-A}}{\mu_0 (1-A)}, \quad (41)$$

we find [22]

$$V_{\nu \pm nk} \sim \left[(\nu \pm nk)^\alpha \right]^{3/[2(1-A)]}, \quad (42)$$

where $\alpha = \mu_0 (1-A)/k$.

We then take the continuum limit of these expressions by expanding $\psi_\nu \approx \psi(\nu)$ as a Taylor expansion in small k/ν . For the particular factor ordering chosen above, the large scale continuum limit of the Hamiltonian constraint reads [22]:

$$\begin{aligned} \lim_{k/\nu \rightarrow 0} \epsilon_{ijk} \text{tr} \left(\hat{h}_i \hat{h}_j \hat{h}_i^{-1} \hat{h}_j^{-1} \hat{h}_k \left[\hat{h}_k^{-1}, \hat{V} \right] \right) |\Psi\rangle &\sim \\ \frac{-36i}{1-A} \alpha^{3/[2(1-A)]} k^3 \sum_\nu \nu^{(1+2A)/[2(1-A)]} &\left[\frac{d^2 \psi}{d\nu^2} + \frac{1+2A}{1-A} \frac{1}{\nu} \frac{d\psi}{d\nu} + \frac{(1+2A)(4A-1)}{(1-A)^2} \frac{1}{4\nu^2} \psi(\nu) \right] |\nu\rangle. \end{aligned} \quad (43)$$

One can easily confirm that one obtains the same continuum limit for the Wheeler-de Witt equation, only for the choice $A = -1/2$ [22], in which case the Wheeler-de Witt equation reads [22]

$$\lim_{k/\nu \rightarrow 0} \mathcal{C}_g |\Psi\rangle = \frac{72}{\kappa^2 \hbar \gamma^3} \left(\frac{\kappa \gamma \hbar}{6} \right)^{3/2} \sum_{\nu} \frac{d^2 \psi}{d\nu^2} |\nu\rangle . \quad (44)$$

Thus, there is only one lattice refinement model, namely $\tilde{\mu} = \mu_0 \mu^{-1/2}$, with a non-ambiguous continuum limit.

In conclusion, phenomenological and consistency requirements lead to a particular lattice refinement model, implying that LQC predicts a unique factor ordering of the Wheeler-de Witt equation in its continuum limit. Alternatively, demanding that factor ordering ambiguities disappear in LQC at the level of Wheeler-de Witt equation leads to a unique choice for the lattice refinement model.

3.4. Numerical techniques in solving the Hamiltonian constraint

Lattice refinement leads to new dynamical difference equations, which being of a non-uniform step-size, imply technical complications. More precisely, the information needed to calculate the wave-function at a given lattice point is not provided by previous iterations. This becomes clear in the case of two-dimensional wave-functions, as for instance in the study of Bianchi models or black hole interiors. I will present below a method [24] based on Taylor expansions that can be used to perform the necessary interpolations with a well-defined and predictable accuracy.

For a one-dimensional difference equation defined on a varying lattice, the Hamiltonian constraint can be mapped onto a fixed lattice simply by a change of basis [24]. This method is however not useful for the two-dimensional case, where the Hamiltonian constraint is a difference equation on a varying lattice [17]:

$$\begin{aligned} & C_+ (\mu, \tau) [\Psi_{\mu+2\delta_\mu, \tau+2\delta_\tau} - \Psi_{\mu-2\delta_\mu, \tau+2\delta_\tau}] \\ & + C_0 (\mu, \tau) [(\mu + 2\delta_\mu) \Psi_{\mu+4\delta_\mu, \tau} - 2(1 + 2\gamma^2 \delta_\mu^2) \mu \Psi_{\mu, \tau} + (\mu - 2\delta_\mu) \Psi_{\mu-4\delta_\mu, \tau}] \\ & + C_- (\mu, \tau) [\Psi_{\mu-2\delta_\mu, \tau-2\delta_\tau} - \Psi_{\mu+2\delta_\mu, \tau-2\delta_\tau}] = \frac{\delta_\tau \delta_\mu^2}{\delta^3} \mathcal{H}_\phi \Psi_{\mu, \tau} , \end{aligned} \quad (45)$$

with

$$C_\pm \equiv 2\delta_\mu \left(\sqrt{|\tau \pm 2\delta_\tau|} + \sqrt{|\tau|} \right) , \quad (46)$$

$$C_0 \equiv \sqrt{|\tau + \delta_\tau|} - \sqrt{|\tau - \delta_\tau|} , \quad (47)$$

where δ_μ and δ_τ denote the step-sizes along the μ and τ directions, respectively.

In the case of lattice refinement, δ_μ and δ_τ are decreasing functions of μ and τ , respectively, and the data needed to calculate the value of the wave-function at a particular lattice site are not given by previous iterations, as it is illustrated in Fig.2. We propose [24] to use Taylor expansions to calculate the necessary data points. Let us assume that the matter Hamiltonian acts diagonally on the basis states of the wave-function, namely

$$\hat{\mathcal{H}}_\phi |\Psi\rangle \equiv \hat{\mathcal{H}}_\phi \sum_{\mu, \tau} \Psi_{\mu, \tau} |\mu, \tau\rangle = \sum_{\mu, \tau} \mathcal{H}_\phi \Psi_{\mu, \tau} |\mu, \tau\rangle . \quad (48)$$

Given a function evaluated at three (non-collinear) coordinates, the Taylor approximation to the value at a fourth position is

$$\begin{aligned} f(x_4, y_4) &= f(x_2, y_2) + \delta_{42}^x \frac{\partial f}{\partial x} \Big|_{x_2, y_2} + \delta_{42}^y \frac{\partial f}{\partial y} \Big|_{x_2, y_2} \\ &+ \mathcal{O} \left((\delta_{42}^x)^2 \frac{\partial^2 f}{\partial x^2} \Big|_{x_2, y_2} \right) + \mathcal{O} \left((\delta_{42}^y)^2 \frac{\partial^2 f}{\partial y^2} \Big|_{x_2, y_2} \right) , \end{aligned} \quad (49)$$

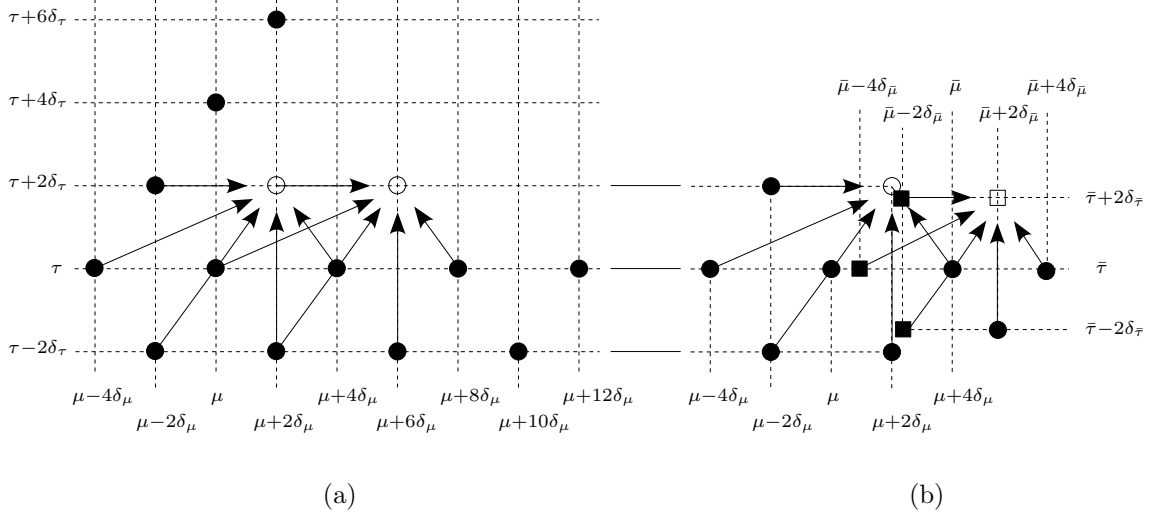


Figure 2. (a) For the fixed lattice case the two-dimensional wave-function can be calculated, given suitable initial conditions (solid circles). (b) In the case of a refining lattice, the data needed to calculate the value of the wave-function at a particular lattice site (open square) is not given by previous iterations (solid squares) [24].

where the Taylor expansion is taken about the position (x_2, y_2) , we have defined $\delta_{ij}^x \equiv x_i - x_j$ and $\delta_{ij}^y \equiv y_i - y_j$, and the differentials can be approximated using

$$f(x_1, y_1) = f(x_2, y_2) + \delta_{12}^x \frac{\partial f}{\partial x} \Big|_{x_2, y_2} + \delta_{12}^y \frac{\partial f}{\partial y} \Big|_{x_2, y_2} + \dots, \quad (50)$$

$$f(x_3, y_3) = f(x_2, y_2) + \delta_{32}^x \frac{\partial f}{\partial x} \Big|_{x_2, y_2} + \delta_{32}^y \frac{\partial f}{\partial y} \Big|_{x_2, y_2} + \dots, \quad (51)$$

where the dots indicate higher order terms.

For slowly varying wave-functions, linear approximation is very accurate and higher order terms in Taylor expansion can only improve the accuracy by $10^{-2}\%$ [24]. This method can be applied in any lattice refinement model, while its accuracy can be estimated.

By using this Taylor expansions method, we confirmed [24] numerically the stability criterion of the Schwarzschild interior, found [17] earlier using a von Neumann analysis, and investigated [24] the way that lattice refinement modifies the stability properties of the system.

Let me now show how the underlying discreteness of space-time leads to a twist [24] in the wave-functions, for both a fixed and a varying lattice model. In the case of a constant lattice, an initially centred Gaussian will move to larger μ , as τ is increased [24]. However, for regions where the lattice discreteness is important, this no longer holds, and the value at one lattice point introduces a non-zero component to the value at a lattice point with larger μ coordinate, i.e., the wave-function moves to larger μ . This implies the existence of some induced rotation on the wave-function due to the underlying discreteness of the space-time. Including lattice refinement, this effect persists [24]. In other words, there is no motion for $\tau \gg \delta_\tau$, while in the case of lattice refinement this requirement is reached for lower τ , since δ_τ reduces as τ increases. As the wave-function moves into a region in which the discreteness of the lattice is important, a motion will be induced, as in the constant lattice case.

4. Anisotropic LQC

Various aspects of anisotropic cosmologies have been studied within LQC in the past [17, 25], however the first full and consistent quantisation of a Bianchi I cosmology (the simplest of anisotropic cosmological models) was achieved recently in Ref. [26]. Moreover, the link back to the underlying full LQG theory has been strengthened, by considering the flux of the triads through surfaces consistent with the Bianchi I anisotropic case. With the quantisation of the Bianchi I model under control, it is possible to ask whether LQC features, obtained within the context of isotropic Friedmann-Lemaître-Robertson-Walker cosmology, are robust, at least with respect to this limited extension of the symmetries of the system. Bianchi type I models, apart their simplicity, they present a particular interest for space-like singularities within the full LQG theory. Following the same vein as for the isotropic case, a massless scalar field plays the rôle of an internal time parameter. In the absence of such a field, physical evolution can be found by constructing families of unitarily related partial observables, parametrised by geometry degrees of freedom [27].

4.1. Unstable Bianchi I LQC

Studying stability conditions of the full Hamiltonian constraint equation describing the quantum dynamics of the diagonal Bianchi I model, I will show [28] that there is robust evidence of an instability in the explicit implementation of the difference equation. On the one hand, such a result may question the choice of the quantisation approach, the model of lattice refinement, and/or the rôle of ambiguity parameters. On the other hand, one may argue that such an instability may not be necessarily a problem since it might be that unstable trajectories are explicitly removed by the physical inner product.

Consider the difference equation arising from the loop quantisation of the Bianchi I model [26]:

$$\begin{aligned} \partial_T^2 \Psi(\lambda_1, \lambda_2, \nu; T) = & \frac{\pi G}{2} \sqrt{\nu} \left[(\nu + 2) \sqrt{\nu + 4} \Psi_4^+(\lambda_1, \lambda_2, \nu; T) - (\nu + 2) \sqrt{\nu} \Psi_0^+(\lambda_1, \lambda_2, \nu; T) \right. \\ & \left. - (\nu - 2) \sqrt{\nu} \Psi_0^-(\lambda_1, \lambda_2, \nu; T) + (\nu - 2) \sqrt{\nu - 4} \Psi_4^-(\lambda_1, \lambda_2, \nu; T) \right], \end{aligned} \quad (52)$$

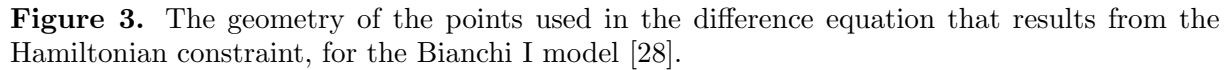
where

$$\begin{aligned} \Psi_4^+(\lambda_1, \lambda_2, \nu; T) &= \sum_{i \neq j=(0,1,2)} \Psi(a_i \lambda_1, a_j \lambda_2, \nu + 4; T) \\ \Psi_4^-(\lambda_1, \lambda_2, \nu; T) &= \sum_{i \neq j=(-3,-2,0)} \Psi(a_i \lambda_1, a_j \lambda_2, \nu - 4; T) \\ \Psi_0^+(\lambda_1, \lambda_2, \nu; T) &= \sum_{i \neq j=(-1,0,1)} \Psi(a_i \lambda_1, a_j \lambda_2, \nu; T) \\ \Psi_0^-(\lambda_1, \lambda_2, \nu; T) &= \sum_{i \neq j=(-2,0,3)} \Psi(a_i \lambda_1, a_j \lambda_2, \nu; T), \end{aligned} \quad (53)$$

$$\begin{aligned} a_{-3} &\equiv \left(\frac{\nu - 4}{\nu - 2} \right), \quad a_{-2} \equiv \left(\frac{\nu - 2}{\nu} \right), \quad a_{-1} \equiv \left(\frac{\nu}{\nu + 2} \right), \\ a_0 &\equiv 1, \quad a_1 \equiv \left(\frac{\nu + 4}{\nu + 2} \right), \quad a_2 \equiv \left(\frac{\nu + 2}{\nu} \right), \quad a_3 \equiv \left(\frac{\nu}{\nu - 2} \right). \end{aligned} \quad (54)$$

Note that $\lambda_1, \lambda_2, \lambda_3$ are related to the volume of the fiducial cell \mathcal{V} by

$$\hat{V} \Psi(\lambda_1, \lambda_2, \nu) = 2\pi |\gamma| \sqrt{\Delta} |\nu| l_{\text{pl}}^3 \Psi(\lambda_1, \lambda_2, \nu), \quad (55)$$


$$\nu = 2\lambda_1\lambda_2\lambda_3 \ , \quad (56)$$

Following the standard von Neumann stability analysis, we decompose the solutions of the difference equation into Fourier modes and look for growing modes. Considering the ansatz [28]

$$\Psi(\lambda_1, \lambda_2, \nu) = T(\lambda_1) \exp(i(\omega\lambda_2 + \chi\nu)) \quad , \quad (57)$$

where we have chosen the λ_1 direction to be the direction in which the $\nu + 4$ plane is *evolved*,

the difference equation becomes

$$\begin{aligned}
e^{4\chi i} \sum_{i \neq j=(0,1,2)} T(a_i \lambda_1) e^{i(\omega a_j \lambda_2 + \chi \nu)} &= \sqrt{\frac{\nu}{\nu+4}} \sum_{i \neq j=(-1,0,1)} T(a_i \lambda_1) e^{i(\omega a_j \lambda_2 + \chi \nu)} \\
&+ \left(\frac{\nu-2}{\nu+2} \right) \sqrt{\frac{\nu}{\nu+4}} \sum_{i \neq j=(-2,0,3)} T(a_i \lambda_1) e^{i(\omega a_j \lambda_2 + \chi \nu)} \\
&- \left(\frac{\nu-2}{\nu+2} \right) \sqrt{\frac{|\nu-4|}{\nu+4}} e^{-4\chi i} \sum_{i \neq j=(-3,-2,0)} T(a_i \lambda_1) e^{i(\omega a_j \lambda_2 + \chi \nu)}.
\end{aligned} \tag{58}$$

As it has been explicitly shown in Ref. [28], expanding in terms of small $1/\nu$, the difference equation can be written in the form of a vector equation as

$$M_1 \bar{T}_3 = M_2 \bar{T}_2, \tag{59}$$

where we have defined the vectors

$$\bar{T}_i = \begin{bmatrix} T(a_i \lambda_1) \\ T(a_{i-1} \lambda_1) \\ T(a_{i-2} \lambda_1) \\ T(a_{i-3} \lambda_1) \\ T(a_{i-4} \lambda_1) \\ T(a_{i-5} \lambda_1) \end{bmatrix} \quad \text{for } i = 2, 3 \tag{60}$$

and the matrices

$$M_1 = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} B & C & D & E & F & G \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{61}$$

The condition for stability can be written as follows: If

$$\max |\tilde{\lambda}| \leq 1 \quad \forall \quad \omega \text{ and } \chi, \tag{62}$$

where $\tilde{\lambda}$ are the eigenvalues of the matrix $(M_1)^{-1} M_2$, the amplitude $T(a_3 \lambda_1)$ is less than that of previous points, in other words, the difference equation is stable. Let us define the parameter $\Lambda = \omega \lambda_2 / \nu$. The presence of an instability in the difference equation has been demonstrated [28] in several ways: (i) there is a particular set of critical modes, $\Lambda = (2n-1)\pi/2$, with $n \in \mathbb{Z}$, for which the system is unstable; (ii) in the large ν limit, the system is unstable for the modes $\Lambda = \pi/4$ and $\chi = 0$; (iii) the system is unstable for a general ν , for modes that approach the critical value.

In conclusion, the difference equation, Eq. (52), is unconditionally unstable, in the sense that there is no region of $(\lambda_1, \lambda_2, \nu)$ in which Eq. (52) is stable.

4.2. Lattice refinement from isotropic embedding of anisotropic cosmology

Given a consistent quantum anisotropic model, one can find [29] isotropic states for which the discrete step-size of the isotropically embedded Hamiltonian constraint is not necessarily that of the $p^{-1/2}$ (*new quantisation*) step-size. The choice of different embeddings has important consequences for the precise form of discretisation in the isotropic sub-system. In this sense, lattice refinement could be interpreted as being due to the degrees of freedom that are absent in the isotropic model [29].

In the standard approach, isotropic states are taken to be those in which the three scale factors are $\lambda_1 = \lambda_2 = \lambda_3$, or more precisely, defining the volume of the state as $\nu = 2\lambda_1\lambda_2\lambda_3$ to eliminate one of the directions (λ_3), the map

$$|\Psi(\lambda_1, \lambda_2, \nu)\rangle \rightarrow \left| \sum_{\lambda_1, \lambda_2} \Psi(\lambda_1, \lambda_2, \nu) \right\rangle \equiv |\Psi(\nu)\rangle, \quad (63)$$

produces isotropic states. Working with the three scale factors λ_i , one can show [29] that there is an ambiguity in exactly what the volume of such an isotropic state is. Consider the state

$$|\tilde{\Psi}(\lambda_1, \lambda_2, \lambda_3)\rangle \equiv \frac{1}{A} \left[|\Psi(\lambda_1, \lambda_2, \lambda_3)\rangle + |\Psi(\lambda_3, \lambda_1, \lambda_2)\rangle + |\Psi(\lambda_2, \lambda_3, \lambda_1)\rangle \right], \quad (64)$$

with the restriction

$$\langle \Psi(\lambda_1, \lambda_2, \lambda_3) | \Psi(\lambda_3, \lambda_1, \lambda_2) \rangle = \langle \Psi(\lambda_1, \lambda_2, \lambda_3) | \Psi(\lambda_2, \lambda_3, \lambda_1) \rangle = \langle \Psi(\lambda_3, \lambda_1, \lambda_2) | \Psi(\lambda_2, \lambda_3, \lambda_1) \rangle,$$

on the anisotropic states. The expectation values of the scale factors along each direction of such a state are

$$\langle \hat{\lambda}_i \rangle = \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}. \quad (65)$$

The measured scale factor is equal in each direction and is given by the average of the scale factors of the underlying, anisotropic states. However, the measured volume of such a state is just $\nu = 2\lambda_1\lambda_2\lambda_3$, which is *not* the cube of the measured scale factor. Thus, while it is the eigenvalue of the anisotropically defined volume operator, it is not necessarily what we would measure as the volume. Essentially, the reason for this is that while the scale factors λ_i are measured to be equal in each direction, they are not eigenvalues of the state, *i.e.*, $\hat{\lambda}_i |\tilde{\Psi}(\lambda_1, \lambda_2, \lambda_3)\rangle \neq \lambda_i |\tilde{\Psi}(\lambda_1, \lambda_2, \lambda_3)\rangle$. However, both the average and the product (*i.e.*, the volume) of the scale factors are eigenvalues. It is this ambiguity, that leads to the possibility of deviations from the standard isotropic case.

In conclusion, the difference between the two procedures is essentially due to what one considers to be more fundamental, the volume of the underlying states (ν) or the measured volume of the symmetric state ($\langle \lambda \rangle^3$), which are not necessarily equal. Choosing ν leads to the *new quantised* Hamiltonian of isotropic cosmology, while choosing $\langle \lambda \rangle^3$ results in some kind of different *lattice refinement*. This *lattice refinement* is significantly more complicated than the single power law behaviour, p^4 , usually considered.

5. Conclusions

Loop Quantum Gravity proposes a method of quantising gravity in a background independent, non-perturbative way. Quantum gravity is essential when curvature becomes large, as for example in the early stages of the evolution of the universe. Applying LQG in a cosmological context leads to Loop Quantum Cosmology which is a symmetry reduction of the infinite dimensional phase space of the full theory, allowing us to study certain aspects of the theory analytically. The discreteness of spatial geometry, a key element of the full theory, leads to successes in LQC which do not hold in the Wheeler-de Witt quantum cosmology.

By studying phenomenological consequences of LQC we can get some useful insight about the full LQG theory and get an answer to some long-standing questions in early universe cosmology. Here, I have briefly described some of the phenomenological aspects of LQC.

Acknowledgments

It is a pleasure to thank the organisers of the First Mediterranean Conference on Classical and Quantum Gravity (MCCQG) (<http://www.phy.olemiss.edu/mccqg/>), for inviting me to give this talk, in the beautiful island of Crete in Greece. This work is partially supported by the European Union through the Marie Curie Research and Training Network *UniverseNet* (MRTN-CT-2006-035863).

References

- [1] Guth A 1981 *Phys. Rev. D* **23** 347
- [2] Sakellariadou M 2008 *Lect. Notes Phys.* **738** 359
- [3] Piran T 1986 *Phys. Lett. B* **181** 238
- [4] Goldwirth D 1991 *Phys. Rev. D* **43** 3204
- [5] Calzetta E and Sakellariadou M 1992 *Phys. Rev. D* **45** 2802
- [6] Calzetta E and Sakellariadou M 1993 *Phys. Rev. D* **47** 3184
- [7] Germani C, Nelson W and Sakellariadou M 2007 *Phys. Rev. D* **76** 043529
- [8] Gibbons G and Turok N 2008 *Phys. Rev. D* **77** 063516
- [9] Ashtekar A and Sloan D 2009 Loop quantum cosmology and slow roll inflation (*Preprint* 0912.4093)
- [10] Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)
- [11] Thiemann T 2007 *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press)
- [12] Bojowald M 2005 *Living Rev. Rel.* **8** 11
- [13] Ashtekar A and Lewandowski J 1997 *Class. & Quant. Grav.* **14** A55
- [14] Ashtekar A, Pawłowski T and Singh P 2006 *Phys. Rev. D* **74** 084003
- [15] Vandersloot K 2005 *Phys. Rev. D* **71** 103506
- [16] Rosen J, Jung J H and Khanna G 2006 *Class. Quant. Grav.* **23** 7075
- [17] Bojowald M, Cartin D and Khanna G 2007 *Phys. Rev. D* **76** 064018
- [18] Sakellariadou M 2009 *J. Phys. Conf. Ser.* **189** 012035
- [19] Nelson W and Sakellariadou M (2007) *Phys. Rev. D* **76** 104003
- [20] Nelson W and Sakellariadou M 2007 *Phys. Rev. D* **76** 044015
- [21] Corichi A and Singh P 2008 *Phys. Rev. D* **78** 024034
- [22] Nelson W and Sakellariadou M 2008 *Phys. Rev. D* **78** 024006
- [23] Ashtekar A, Pawłowski T and Singh P 2006 *Phys. Rev. D* **74**, 084003
- [24] Nelson W and Sakellariadou M 2008 *Phys. Rev. D* **78** 024030
- [25] Chiou D W and Vandersloot K 2007 *Phys. Rev. D* **76** 084015
- [26] Ashtekar A and Wilson-Ewing E 2009 *Phys. Rev. D* **79** 083535
- [27] Martin-Benito M, Mena Marugan G A and Pawłowski T 2009 *Phys. Rev. D* **80** 084038
- [28] Nelson W and Sakellariadou M 2009 *Phys. Rev. D* **80** 063521
- [29] Nelson W and Sakellariadou M 2009 Lattice Refining Loop Quantum Cosmology from an Isotropic Embedding of Anisotropic Cosmology (*Preprint* 0906.0292)